

Robust model predictive control with disturbance invariant sets

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Abstract—This paper proposes a robust model predictive control scheme for nonlinear systems with state and input constraints and unknown but bounded disturbances. A standard nominal model predictive control problem with tightened constraints is solved online, and its solution defines the nominal trajectory. An ancillary control law is determined off-line which keeps the trajectories of the error system in a disturbance invariant set. Thus, the evolution of original nonlinear system lies in the disturbance invariant set centered along the nominal trajectory. Furthermore, it is shown that both feasibility and stability of the closed-loop system are guaranteed if the standard nominal optimization problem is initially feasible.

I. INTRODUCTION

Model predictive control (MPC) or receding horizon control (RHC) is a class of optimization based control methods in which a control sequence is determined by optimizing a finite horizon cost at each sampling instant, based on an explicit process model and state measurement. The first control action of the optimal sequence is applied to the plant. At the next sampling instant, the optimization problem is solved again using new measurements, and the control input is updated. Due to its ability to handle constraints on inputs and states, this control method has received much interest in both academic community and industrial society over the last 30 years [1, 2]. By introducing a so-called stability constraint and appropriately computing a terminal penalty term, nominal stability issues are well-addressed [1, 3]. Although it has been proven that an MPC controller inherently has some degree of robustness [4–6], uncertainties in the process model can destabilize a system with nominally asymptotically stable MPC controller [7, 8].

In order to achieve robustness, the controller must stabilize the system for all possible realizations of the uncertainty. An intuitive way is to solve a min-max optimization problem online when disturbances and/or model mismatch are present [9–13]. In general, these schemes are computationally intractable since the size of the optimization problem required for their solution grows exponentially with the increase of the prediction horizon [10]. The constraint tightening approach, which is introduced in [7, 14, 15], avoids computational complexity by using a nominal prediction model and modifying the constraint sets to achieve robustness. However, the constraint sets will shrink drastically because the “margin”, which retains in each optimization for the

impact of uncertainty on the actual system, will increase with the prolonging of the prediction horizon. For linear discrete-time systems with persistent disturbances, [16] provides a new constraint tightening, tubed based robust MPC scheme, which demonstrates the reduction of online computational burden while the constraint sets are not shrinking drastically. The algorithm utilizes both feedback control law and feedforward control action. The state feedback control law is designed offline as a *nominally* stabilizing control policy, which keeps the evolution of the constrained systems in a disturbance invariant set. The feedforward control action is calculated online and steers the nominal system states to the equilibrium. However, the shape of the minimal disturbance invariant set is not a priori known. The results proposed in [16] depend on the linearity property of the considered system class, and have been extended in [17] to some classes of nonlinear discrete-time systems, namely systems with matched nonlinearity and a particular class of piecewise affine systems. The result relies on a parameterized feedback control policy which transforms the considered systems to linear systems.

In this paper we use feedback and feedforward components similar to [16, 17] and propose a robust model predictive controller for general nonlinear systems with persistent disturbances. The result relies on the offline calculation of a *robustly* stabilizing ancillary control law for nonlinear systems, and a convex, disturbance invariant set chosen to be an ellipsoid. The feedforward control action, calculated repeatedly online, generates a nominal trajectory. The pre-determined ancillary control law keeps all admissible trajectories of the uncertain systems in the disturbance invariant set (i.e., in the ellipsoid) centered around the nominal trajectory.

The remainder of the paper is organized as follows. Section 2 defines the problem statement. Section 3 presents the main results, including the construction of a robust invariant set and the calculation of the ancillary control law. The proposed scheme is proven to be robustly stable and feasible. In Section 4, a simulation example is provided to demonstrate the effectiveness of the proposed scheme. Section 5 concludes the paper with a brief summary.

II. PROBLEM SETUP

Consider a system described by a nonlinear ordinary differential equation with bounded disturbances:

$$\dot{x}(t) = f(x(t), u(t), w(t)), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state of the system and $u(t) \in \mathbb{R}^m$ is the control input. The signal $w(t) \in \mathbb{R}^p$ is the exogenous

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disturbance or uncertainty, lying in the compact set

$$\mathbb{W} = \{w(t) \in \mathbb{R}^p \mid \|w(t)\|_\infty \leq w_{max}\},$$

which contains the origin, i.e. $w(t) \in \mathbb{W}, \forall t \geq 0$. The system is subject to constraints on both state and input given by

$$x(t) \in \mathbb{X}, \quad u(t) \in \mathbb{U} \quad \forall t \geq 0, \quad (2)$$

where \mathbb{X} and \mathbb{U} are compact sets containing the origin. Some fundamental assumptions are stated in the following:

- Assumption 1:* A0) $f(x, u, w) : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^n$ is continuously differential for x, u and w . Furthermore, $f(0, 0, 0) = 0$, thus $0 \in \mathbb{R}^n$ is an equilibrium of the system.
- A1) System (1) has a unique solution for any initial condition $x_0 \in \mathbb{X}$ and any piecewise right-continuous input function $u(\cdot) : [0, T_p] \rightarrow \mathbb{U}$ and $w(\cdot) : [0, T_p] \rightarrow \mathbb{W}$;
- A2) $\mathbb{U} \subset \mathbb{R}^m$ is compact, $\mathbb{X} \subseteq \mathbb{R}^n$ is connected and the point $(0, 0, 0)$ is contained in the interior of $\mathbb{X} \times \mathbb{U}$.

For the nominal system, we have $w(t) = 0$. Thus, it is described by

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t), 0). \quad (3)$$

For simplification we denote (3) as $\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t))$. The error between the actual and nominal state defined as $v = x - \bar{x}$, hence satisfies

$$\dot{v} = f(x, u, w) - f(\bar{x}, \bar{u}). \quad (4)$$

In the following we use for (4) the expression *error system*. We will design a control signal with both nominal controller and pre-derived feedback terms as follows:

$$u = \bar{u} + \pi(x - \bar{x}), \quad (5)$$

where $\pi(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a control law, linear or nonlinear. The nominal controller \bar{u} is obtained by the solution of nominal optimal control problem with tightened constraints. We use the nominal control to generate a nominal trajectory, and the ancillary control law $\pi(x - \bar{x})$ to keep all solutions of the error system (4) in an invariant set centered on the nominal trajectory.

Before proceeding with the main results of the paper it is necessary to define the concept of robust control invariant sets and some set operations [18].

Definition 1: (Robust control invariant set) The set $\Omega \subset \mathbb{R}^n$ is a robust control invariant set for the error system (4) if and only if there exists a feedback control law $u = \kappa(x)$ such that $\forall v(t_0) \in \Omega$ and $\forall w \in \mathbb{W}$, the trajectories $v(t)$ remain in Ω for all $t \geq 0$.

Definition 2: Consider two sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$, then the Pontryagin difference set is defined as

$$\mathcal{A} \sim \mathcal{B} = \{x \in \mathbb{R}^n \mid x + y \in \mathcal{A}, \forall y \in \mathcal{B}\},$$

Similarly, the addition set is defined as

$$\mathcal{A} \oplus \mathcal{B} = \{x + y \mid x \in \mathcal{A}, y \in \mathcal{B}\}.$$

Definition 3: The multiplication of a set \mathcal{B} by a matrix A denotes a mapping of all its elements

$$A\mathcal{B} = \{c \mid \exists b \in \mathcal{B}, c = Ab\},$$

III. ROBUST MODEL PREDICTIVE CONTROLLER

In this section we present a robust model predictive controller for nonlinear systems that has two components: a nominal controller that generates a nominal open-loop control input and a nominal state trajectory (calculated online) and the ancillary control law K (calculated offline) which aims at steering the trajectories of the error system (4) to the origin, i.e. the trajectory of system (1) to the nominal trajectories.

A. Nominal control input

The nominal open-loop optimal control problem is subject to the nominal dynamics (3), i.e. no disturbances are present. Furthermore, it is subject to tighter constraints than the original constraints introduced in (2) in order to guarantee satisfaction of the original constraints. As in [16], the model predictive controller we proposed is based on the repeated online solution of an optimal control problem in which the initial state of the nominal model is a decision variable.

For the current state $x(t_k)$, the nominal control problem which is solved *online* is formulated as follows:

$$\mathbf{P}(x) : \min_{\bar{x}(t_k), \bar{u}(\cdot)} J(\bar{x}(t_k), \bar{u}(\cdot)), \quad (6)$$

subject to

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}), \quad (7a)$$

$$x(t_k) - \bar{x}(t_k) \in \Omega, \quad (7b)$$

$$\bar{x}(t_k + \tau; \bar{x}(t_k), t_k) \in \mathbb{X}_0, \quad \tau \in [0, T_p], \quad (7c)$$

$$\bar{u}(t_k + \tau; \bar{x}(t_k), t_k) \in \mathbb{U}_0, \quad \tau \in [0, T_p], \quad (7d)$$

$$\bar{x}(t_k + T_p; \bar{x}(t_k), t_k) \in \mathbb{X}_f, \quad (7e)$$

where $\mathbb{X}_0 \doteq \mathbb{X} \sim \Omega$, $\mathbb{U}_0 \doteq \mathbb{U} \sim K\Omega$, $\mathbb{X}_f \subset \mathbb{X} \sim \Omega$, and

$$J(\bar{x}(t_k), \bar{u}(\cdot)) = \int_{t_k}^{t_k + T_p} \bar{x}^T(\tau; \bar{x}(t_k), t_k) Q \bar{x}(\tau; \bar{x}(t_k), t_k) d\tau + \bar{x}^T(\tau; \bar{x}(t_k), t_k) R \bar{u}(\tau; \bar{x}(t_k), t_k) d\tau + F(\bar{x}(t_k + T_p)). \quad (8)$$

Here T_p is the prediction horizon, $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ are positive definite weighting matrices. The set Ω is a disturbance invariant set for the error system which will be introduced in detail later. The pair $(\bar{x}^*(t_k), \bar{u}^*(\cdot))$ denotes the optimal solution to the open-loop optimal control problem $\mathbf{P}(x)$, and $\bar{x}^*(\cdot; \bar{x}^*(t_k), t_k)$ is the predicted trajectory of (3) starting from the state $\bar{x}^*(t_k)$ at time t_k and driven by a given open-loop input function $\bar{u}^*(\cdot; \bar{x}^*(t_k), t_k)$.

The applied nominal control input is defined as

$$\bar{u}(\tau) = \bar{u}^*(\tau; \bar{x}^*(t_k), t_k), \quad \tau \in [t_k, t_k + \delta), \quad (9)$$

The set \mathbb{X}_f is a neighborhood of the origin which is a level set of a positive definite function $F(\cdot)$. Moreover, \mathbb{X}_f and $F(x)$ satisfy the following terminal conditions [3, 19]:

B0) $\mathbb{X}_f \subseteq \mathbb{X}_0$,

B1) $\kappa(\bar{x}) \in \mathbb{U}_0$, for all $\bar{x} \in \mathbb{X}_f$,

B2) $F(x)$ satisfies inequality,

$$\frac{\partial F(x)}{\partial x} f(\bar{x}, \kappa(\bar{x})) + \bar{x}^T Q \bar{x} + \kappa(\bar{x})^T R \kappa(\bar{x}) \leq 0, \quad \forall \bar{x} \in \mathbb{X}_f \quad (10)$$

where $\bar{u} = \kappa(\bar{x})$ is a feasible control law.

We assume that for the nominal system (3), there exists a locally asymptotically stabilizing controller $\bar{u} = \kappa(\bar{x})$, a terminal region \mathbb{X}_f and a continuously differentiable, positive definite function $F(\bar{x})$ that satisfy (10) for $\forall \bar{x} \in \mathbb{X}_f$ [3]. Since (10) holds, \mathbb{X}_f is invariant for the nominal system (3) controlled by $u = \kappa(x)$. It is well-known that MPC stabilizes the nominal system [1, 3] if the terminal conditions B0-B2 are satisfied. Furthermore, the optimal value function J^* satisfies:

$$\begin{aligned} & J^*(\bar{x}(t + \delta; \bar{x}(t), t)) - J^*(\bar{x}(t)) \\ & \leq -\bar{x}(t)^T Q \bar{x}(t) - \bar{u}(t)^T R \bar{u}(t), \end{aligned} \quad (11)$$

where $\delta = t_{k+1} - t_k$.

B. Disturbance invariance of nonlinear systems

In this subsection, we present a general result of disturbance invariance of nonlinear systems, which will be required for further consideration.

Lemma 1: Suppose there exists $E(v(t)) \geq 0$, $\lambda > 0$ and $\mu > 0$ such that

$$\frac{d}{dt}E(v(t)) + \lambda E(v(t)) - \mu w^T(t)w(t) \leq 0. \quad (12)$$

Then, the system trajectory starting from $v(t_0) \in \Omega(v)$ will remain in $\Omega(v)$, where

$$\Omega(v) := \{v | E(v(t)) \leq \frac{\mu w_{max}^2}{\lambda}\}. \quad (13)$$

Proof: Multiplying (12) by the $e^{\lambda t}$ yields

$$\frac{d}{dt}(e^{\lambda t}E(v(t))) \leq \mu e^{\lambda t}w^T(t)w(t). \quad (14)$$

Integrating (14) from t_0 to t leads to

$$E(v(t)) \leq e^{-\lambda(t-t_0)}E(v(t_0)) + \mu e^{-\lambda t} \int_{t_0}^t e^{\lambda s} w^T(s)w(s) ds.$$

Due to $\|w\|_\infty \leq w_{max}$, we have

$$\begin{aligned} E(v(t)) & \leq e^{-\lambda(t-t_0)}E(v(t_0)) + \mu w_{max}^2 e^{-\lambda t} \int_{t_0}^t e^{\lambda s} ds \\ & = e^{-\lambda(t-t_0)} \left(E(v(t_0)) - \frac{\mu w_{max}^2}{\lambda} \right) + \frac{\mu w_{max}^2}{\lambda}. \end{aligned}$$

In virtue of $v(t_0) \in \Omega(v)$, we have $E(v(t_0)) \leq \frac{\mu w_{max}^2}{\lambda}$ for all $t \geq t_0$. \square

Remark 3.1: For linear systems $\dot{x} = Ax$, or linear selectionable differential inclusions $\dot{x} \in A(x)$ where $A(x) = \{v | v = Cx, C \in \mathcal{C}\}$ and $\mathcal{C} \subset \mathbb{R}^{n \times n}$ is a convex set, asymptotic stability of the origin is equivalent to the existence of a piecewise linear positive definite function $S(x)$ and a positive scalar λ which satisfy the inequality $\dot{S}(x) + \lambda S(x) \leq 0$ for the matrix A or the linear selectionable differential inclusion $A(x)$, respectively [20].

Remark 3.2: Condition (12) shows that, if $E(v)$, λ and μ exist, the error system (4) is exponentially stable with the decay rate λ in the case of vanishing disturbance $\lim_{t \rightarrow \infty} w(t) = 0$

If $E(\cdot)$ is a control Lyapunov function, we can find a control law by solving inequality (12), which guarantees that the nonlinear system stays in the set Ω , if the initial state of the system lies in this set. In general, for nonlinear systems, it is hard to find a control Lyapunov function $E(\cdot)$ and an associated control law such that (12) is satisfied. In the next subsection we provide sufficient (and therefore conservative) conditions for the calculation of a quadratic control Lyapunov function $E(x) = x^T P x$ and an ancillary linear feedback controller $u = Kx$. Note that the ancillary control law is calculated *offline*.

C. Case study: a static ancillary control law

Suppose that for all $x \in \mathbb{X}_0$, $u \in \mathbb{U}_0$ and $w \in \mathbb{W}$ there exists a matrix $G(x, u, w) \in \Sigma$ such that $f(x, u, w) = G(x, u, w) \begin{bmatrix} x & u & w \end{bmatrix}^T$, where $\Sigma \subseteq \mathbb{R}^{(n+m+p)}$ is a polytopic linear differential inclusion (PLDI) of the nonlinear system (1):

$$\Sigma = \text{Co} \{ [A_1 \ B_1 \ B_{w1}], \dots, [A_L \ B_L \ B_{wL}] \}. \quad (16)$$

Here, $[A_i \ B_i \ B_{wi}]$, $i = 1, 2, \dots, L$, are vertex matrices of the set Σ , and L is the number of vertex matrices. Conditions that guarantee existence of the differential inclusion are $f(0, 0, 0) = 0$ and $[\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial u} \ \frac{\partial f}{\partial w}] \in \Sigma$ for all x, u, w , see e.g. [21]. If we can show that the PLDI has some property, then this property also holds for the nonlinear system [21].

Remark 3.3: The nonlinear system (1) and the error system (4) have the same PLDI for $\forall x \in \mathbb{X}_0$, $u \in \mathbb{U}_0$ and $w \in \mathbb{W}$.

Lemma 2: Suppose that there exist matrices $0 < X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{m \times n}$, scalars $\lambda > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} (A_i X + B_i Y)^T + A_i X + B_i Y + \lambda X & B_{wi} \\ * & -\mu I \end{bmatrix} \leq 0, \quad (17)$$

$i = 1, \dots, L.$

where $*$ denotes a submatrix required to enforce symmetry. Then, with $u' = Kv$ and $E(v) = v^T P v$, where $P = X^{-1}$ and $K = YX^{-1}$, Lemma 1 is satisfied for the error system (4).

Proof: Pre-and post-multiplying (17) by $\text{diag}(P, I)$ yields

$$\begin{bmatrix} (A_i + B_i K)^T P + P(A_i + B_i K) + \lambda P & P B_{wi} \\ * & -\mu I \end{bmatrix} \leq 0,$$

which using the Schur complement implies that $\frac{d}{dt}(v^T P v) + \lambda v^T P v - \mu w^T w \leq 0$. Therefore, Lemma 1, holds for the error system (4), $\forall (\bar{x}, \bar{u}, w) \in \mathbb{X}_0 \times \mathbb{U}_0 \times \mathbb{W}$. \square

Strictly speaking, we should know a priori the sets \mathbb{X}_0 and \mathbb{U}_0 since those sets are required to determine the PLDI (16). However both \mathbb{X}_0 and \mathbb{U}_0 depend on the set Ω . This is clearly a contradiction. Therefore, we propose an iterative algorithm for the solution of inequality (17).

Algorithm 1: Step 1. For fixed λ , given $\mathbb{X}_0 \in \mathbb{X}$ and $\mathbb{U}_0 \in \mathbb{U}$, solve inequality (17) and get K and Ω .

Step 2. If $\Omega \in \mathbb{X} \sim \mathbb{X}_0$, and $K\Omega \in \mathbb{U} \sim \mathbb{U}_0$, stop; otherwise set $\mathbb{X}_0 = \mathbb{X}_0 - \rho \mathbb{X}_0 \in \mathbb{X}$ and $\mathbb{U}_0 = \mathbb{U}_0 - \rho \mathbb{U}_0 \in \mathbb{U}$, where $0 < \rho < 1$, go to step 1.

Assumption 2: Suppose that inequality (17) has a feasible solution such that Ω lies in the interior of \mathbb{X} and $K\Omega$ lies in the interior of \mathbb{U} .

Remark 3.4: Clearly, from (13) we know that the choice of a larger λ leads to a smaller volume of the disturbance invariant set and a larger decay rate of the closed-loop system, which implies better control performance. On the other hand, it follows from (17) that the choice of a larger λ leads to a smaller set of feasible solutions (μ, X, Y) . This contradiction motivates us to choose λ as the “almost” largest one which can guarantee the LMI optimization problem to be feasible.

Remark 3.5: As with all differential inclusions schemes for nonlinear control problems, some conservativeness is introduced which shows a potential weakness of the proposed method.

D. Feasibility and stability

Since the disturbances are bounded but not necessarily decaying, the origin is not a steady state of the uncertain system. Hence, the aim of a stabilizing controller is to steer the state to a neighborhood of the origin and to keep the state within it. The set has to be a robust positive invariant set for the closed-loop system and its size depends on the bound on the disturbances. This section proves that for any feasible initial state, the proposed predictive controller steers system (1) to the disturbance invariant set around the origin and remains there for all times.

Due to the repeated solution of the optimal control problem $\mathbf{P}(x)$, we first deduce its feasibility at each time instant.

Theorem 1: (Robust feasibility) Suppose that the optimal control problem $\mathbf{P}(x(t_0))$ is feasible at time $t_0 = 0$. Then, for a sufficiently small sampling time $\delta > 0$, it is feasible for any $t \geq t_0$.

Proof: Assume that at time instant t_0 , the problem $\mathbf{P}(x(t_0))$ is feasible and its solution is the nominal initial state $\bar{x}^*(t_0; x(t_0), t_0)$ and the nominal control input $\bar{u}^*(\cdot; x(t_0), t_0)$. Since $x(t_0) \in \bar{x}^*(t_0) \oplus \Omega$, it follows from the disturbance invariance property of Ω that for system (4) with ancillary control law $\pi(\cdot)$, $x(t_0 + \delta) \in \bar{x}^*(t_0 + \delta; x(t_0), t_0) \oplus \Omega$. Consider the candidate control action

$$\bar{u}^0(\tau) = \begin{cases} \bar{u}^*(\cdot; x(t_0), t_0) & \tau \in [t_0 + \delta, t_0 + T_p], \\ \kappa(\bar{x}^*(\cdot)) & \tau \in [t_0 + T_p, t_0 + T_p + \delta]. \end{cases}$$

The nominal state trajectory associated with $\bar{u}^0(\tau)$ is

$$\bar{x}^0(\tau) = \begin{cases} \bar{x}^*(\cdot; x(t_0), t_0) & \tau \in [t_0 + \delta, t_0 + T_p], \\ \bar{x}(\cdot; \bar{x}^*(t_0 + T_p), t_0 + T_p) & \tau \in [t_0 + T_p, t_0 + T_p + \delta], \end{cases}$$

where $\bar{x}(\cdot; \bar{x}^*(t_0 + T_p), t_0 + T_p)$ is the system evolution with the control input $\kappa(\bar{x}^*(\cdot))$ and the initial state $\bar{x}^*(t_0 + T_p; x(t_0), t_0)$. Since $(\bar{x}^*(\cdot; x(t_0), t_0), \bar{u}^*(\cdot; x(t_0), t_0))$ is feasible for $\mathbf{P}(x(t_0))$, constraints (7c)-(7e) are satisfied. Hence, constraints (7c) and (7d) are satisfied by $\bar{x}^0(\tau)$ and $\bar{u}^0(\tau)$ in the interval $[t + \delta, t + T_p]$. Since $\bar{x}(\cdot; \bar{x}^*(t_0 + T_p), t_0 + T_p) \in \mathbb{X}_f$, it follows from B0)-B2) that the terminal control law $\kappa(\bar{x}(\cdot; \bar{x}^*(t_0 + T_p), t_0 + T_p)) \in U_0$ renders the set \mathbb{X}_f

invariant. Hence, $\kappa(\bar{x}(\cdot; \bar{x}^*(t_0 + T_p), t_0 + T_p)) \in U_0$ satisfies the constraints (7d) and $\bar{x}(\cdot; \bar{x}^*(t_0 + T_p), t_0 + T_p)$ satisfies (7c), $\forall \tau \in [t_0 + T_p, t_0 + T_p + \delta]$. Since $x(t_0 + \delta) \in \bar{x}^*(t_0 + \delta; x(t_0), t_0) \oplus \Omega$, the pair $(\bar{x}^0(\tau), \bar{u}^0(\tau))$ is a feasible solution to $\mathbf{P}(x^*(t_0 + \delta))$. Furthermore, due to the disturbance invariance property of Ω and the feasibility of $\mathbf{P}(\bar{x})$, system (1) controlled by the proposed scheme robustly satisfies the constraints (2). \square

Let us define the candidate Lyapunov function

$$V(x) = J(\bar{x}^*). \quad (18)$$

which is the optimal value function. The following result shows the properties of the candidate Lyapunov function for system (1) under the proposed model predictive control law.

- Lemma 3:* (i) $0 \leq V(x) < +\infty$,
(ii) $V(x) = 0$, for all $x \in \Omega$,
(iii) $V(x(t + \delta)) - V(x(t)) \leq -\bar{x}^*(t)^T Q \bar{x}^*(t) - \bar{u}^*(t)^T R \bar{u}^*(t)$

Proof: (i) follows directly from the definition of $V(\cdot)$.

(ii) Let $x(t)$ be an arbitrary point in Ω . Since $x(t) \in 0 \oplus \Omega$, it follows that $\bar{x}(t)^* = 0$ and $\bar{u}^0(\tau) = 0$, $\forall \tau \geq t$, is a feasible solution to the optimization problem $\mathbf{P}(x(t_0))$. Hence $V(x) \leq J(0) = 0$, which establishes the result.

(iii) Note that $x(t + \delta) \in \bar{x}^*(t + \delta; x(t), t) \oplus \Omega$ such that $(\bar{x}^0(\tau), \bar{u}^0(\tau))$ is feasible for $x(t + \delta)$. Hence, $V(x(t + \delta)) \leq J(\bar{x}^*(t + \delta; x(t), t))$. Furthermore, from (11) $J(\bar{x}^*(t + \delta; x(t), t)) - J(\bar{x}^*(t)) \leq -\bar{x}^*(t)^T Q \bar{x}^*(t) - \bar{u}^*(t)^T R \bar{u}^*(t)$. Since $V(x(t)) = J(\bar{x}^*(t))$, the proposition follows. \square

Definition 4: A system is asymptotically ultimately bounded if the system converges asymptotically to a bounded set [22].

In the following theorem we show that for any feasible initial state, the proposed controller steers system (1) to the disturbance invariant set around origin and remains there for all times. Hence, the system under control is ultimately bounded. Firstly, we proof that Ω is attractive for system (1) under the disturbance $w \in \mathbb{W}$. Then, we show that the controlled system remains there.

Theorem 2: Suppose that Assumptions 1 and 2 are satisfied, and the optimal control problem $\mathbf{P}(x(t_0))$ is feasible at time $t = 0$. Then, for a sufficient small sampling time $\delta > 0$, system (1) is asymptotically ultimately bounded.

Proof: (i) Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that $B_r := \{x \in \mathbb{R}^n, x \notin \Omega, \|x - y\| \leq r, \forall y \in \bar{\Omega}\}$ is a neighborhood of the set Ω , where $\bar{\Omega} = \{x^T P x = \frac{\mu w_{max}^2}{\mu}\}$. Due to the continuity of $V(x)$ and $V(x) > 0$ for all $x \notin \Omega$, there exists a $\beta \in (0, \infty)$ such that $\beta < \min_{\|x - y\| = r} V(x)$ for $\forall y \in \bar{\Omega}$. Define $W_\beta := \{x \in B_r | V(x) \leq \beta\}$, then W_β is entirely contained in the interior of B_r .

(ii) Furthermore, for all $x(t_0) \in \mathbb{X}_0$, there exists a finite time T such that $x(T) \in W_\beta$. This can be proven by contradiction: Assume that $x(t) \notin W_\beta$ for all $t > T$. It

follows that for all $t \geq T$

$$\begin{aligned} V(x(t+\delta)) - V(x(t)) &\leq -\int_t^{t+\delta} \|\bar{x}(\tau)^T Q \bar{x}(\tau)\| d\tau \\ &\leq -\delta \inf \{ \|\bar{x}^T Q \bar{x} | \bar{x} \notin W_\beta \| \} \leq -\delta \gamma \end{aligned}$$

where $x \in \bar{x} \oplus \Omega$ and $\gamma > 0$. By induction, $V(x(t+\delta)) \rightarrow -\infty$ as $t \rightarrow \infty$ if $x(t) \notin W_\beta$ for all t , which contradicts with the fact that $V(\cdot) > 0$. Thus any trajectory of (1) starting from \mathbb{X} enters into W_β in a finite time.

(iii) Because $V(x(t))$ is monotonically nonincreasing and bounded from below by zero, it converges as $t \rightarrow \infty$. Now,

$$\int_{t_0}^t \bar{x}(\tau)^T Q \bar{x}(\tau) d\tau \leq V(x(t)) - V(x(t_0)).$$

Therefore, $\lim_{t \rightarrow \infty} \int_{t_0}^t \bar{x}(\tau)^T Q \bar{x}(\tau) d\tau$ exists and is finite. Since $\bar{x}(t)$ is bounded, $f(\bar{x}, \bar{u})$ is bounded and uniformly in t , for all $t \geq t_0$. Hence, $\bar{x}(t)$ are uniformly continuous in t on $[t_0, \infty)$. Consequently, $\bar{x}^T Q \bar{x}$ is uniformly continuous in t on $[t_0, \infty)$, since $\bar{x}^T Q \bar{x}$ is uniformly continuous in \bar{x} on the compact set $W_\beta \sim \Omega$. Therefore, by Barbalat's Lemma, we conclude that $\bar{x}^T Q \bar{x} \rightarrow 0$ as $t \rightarrow \infty$. In other words, $\bar{x} \rightarrow 0$ and $x \rightarrow \Omega$ as $t \rightarrow \infty$. Thus, Ω is attractive for the nonlinear system (1).

(iv) Since $V(x)$ is continuous at $x \in W_\beta$ and $V(x) = 0$ for all $x \in \Omega$, there exists η ($\eta < r$) such that $\|x - y\| < \eta$, $\forall y \in \bar{\Omega}$, implies $V(x) < \beta$. Consequently,

$$\begin{aligned} \|x(0) - y\| < \eta &\Rightarrow V(x(0)) < \beta \\ \Rightarrow V(x(t)) < \beta &\Rightarrow \|x(t) - y\| < r, \forall y \in \bar{\Omega}. \end{aligned}$$

Thus, the set Ω is robustly exponentially stable for the controlled system (1). \square

Note that stability is guaranteed due to the feasibility of the computed control action at each sampling time. Hence, optimality is not necessary to guarantee stability. Moreover, we derive that at each sampling time we can compute an initial feasible solution based on the solution obtained at the previous sampling time, and this initial state is a hot start for the optimization problem. This allows to relax the computational burden of the optimization problem.

Corollary 1: Suppose that Assumptions 1 and 2 are satisfied and the optimal control problem $\mathbf{P}(x(t_0))$ is feasible at time $t = 0$. Then, for a sufficient small sampling time $\delta > 0$, with the control (5), the closed-loop system is input-to-state stable (ISS).

Proof: By Lemma 2, we know that

$$E(v(t+\delta)) - E(v(t)) \leq -\theta E(v(t)) + \frac{\theta\mu}{\lambda} \|w(t)\|_\infty^2,$$

where $\theta = 1 - e^{-\lambda\delta} > 0$, and $v = x - \bar{x}$.

Define $M(x) = V(x) + E(v)$, with (iii) of Lemma 3, we can get that

$$M(x(t+\delta)) - M(x(t)) \leq -\alpha(x) + \beta(w),$$

where $\alpha(x) = \theta E((x - \bar{x})(t)) - [\bar{x}(t)^T Q \bar{x}(t) + \bar{u}(t)^T R \bar{u}(t)]$ and $\beta(w) = \frac{\theta\mu}{\lambda} \|w(t)\|_\infty^2$. Obviously, $\alpha(\cdot)$ and $\beta(\cdot)$ are \mathcal{K}

functions. Furthermore, in virtue of the definition of $E(\cdot)$ and $V(\cdot)$ we have that

$$0 \leq M(x) \leq \pi(x),$$

where $\pi(\cdot)$ is a \mathcal{K} function. Thus, the optimal cost $M(x)$ is an input-to-state stability Lyapunov function, and the closed-loop system is ISS [23, 24]. \square

Remark 3.6: If the disturbance is decaying with time, Corollary 1 guarantees that the closed-loop system is asymptotically stable, and the origin is its equilibrium point.

IV. ILLUSTRATIVE EXAMPLE

Consider the nonlinear system

$$\begin{cases} \dot{x}_1 &= 0.5x_1 + 0.15x_1^2 + x_2 + 0.6u \\ \dot{x}_2 &= x_1 - 0.2x_2^2 + 0.6u + w. \end{cases}$$

Assume that x_1 and x_2 are measurable. We consider the following input constraints $-2 \leq u \leq 2$. The disturbance is bounded by $w \in \mathbb{W} \doteq \{w \in \mathbb{R} | \|w\|_\infty \leq 0.1\}$. The penalty matrices Q and R are chosen as $Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, $R =$

1. Both the terminal control law and terminal penalty are yielded by the solution of a convex optimization problem, see [25], as $\kappa(x) = [-1.4496 \quad 0.3091]x$, $F(x) = x^T \begin{bmatrix} 13.4654 & -11.0235 \\ -11.0235 & 35.0655 \end{bmatrix} x$. The robust invariant set is

$$\Omega = \{x | x^T P x < 1\} \text{ with } P = 10^{-3} \times \begin{bmatrix} 2.3907 & 1.7277 \\ 1.7277 & 1.2950 \end{bmatrix}.$$

The ancillary control law, given by the feedback matrix $K = [-24.9562 \quad -18.7641]$, guarantees that the set Ω is a robust invariant set for the error system (4). The horizon length of the optimization problem is $T_p = 18$ and the sampling time δ is 0.075. Figure 1 shows the states of the considered system with the disturbance $w = 0.095$. The dashed line shows the trajectory of the nominal system, and the solid line shows the trajectory of the actual system. As can be seen, the trajectories of the actual system under persistent disturbance remain in the ‘‘robust invariant sets’’ around the nominal trajectory, and $\bar{x} = 0$ while the system state $x(t) \in 0 \oplus \Omega$. Furthermore, the system state remains in the set $0 \oplus \Omega$.

V. CONCLUSION

A robustly stabilizing model predictive control (MPC) algorithm with guaranteed recursive feasibility is developed for state and input constrained nonlinear systems with persistent disturbance. The control signal is specified in terms of both nominal control action and ancillary control law. The optimal control problem that is solved online includes the initial state of the model as a decision variable. The ancillary control law is designed to maintain the state of the error systems within a prescribed ellipsoid in the presence of unknown but bounded disturbances, and the nominal control action drives the center of these ellipsoids to a desired reference state. The results are illustrated by a numerical example.

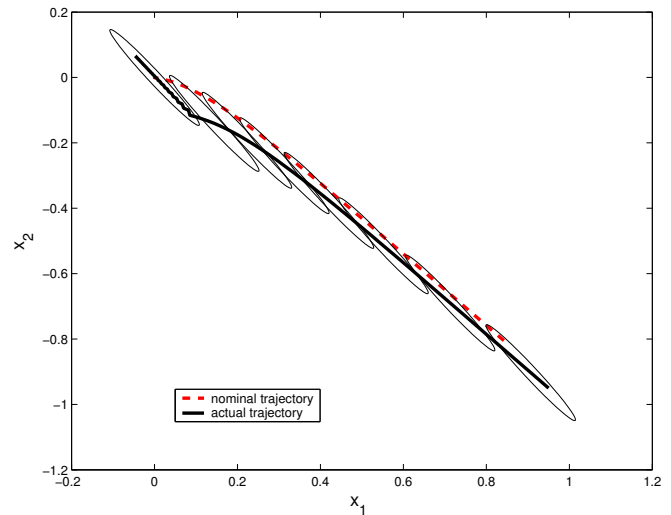
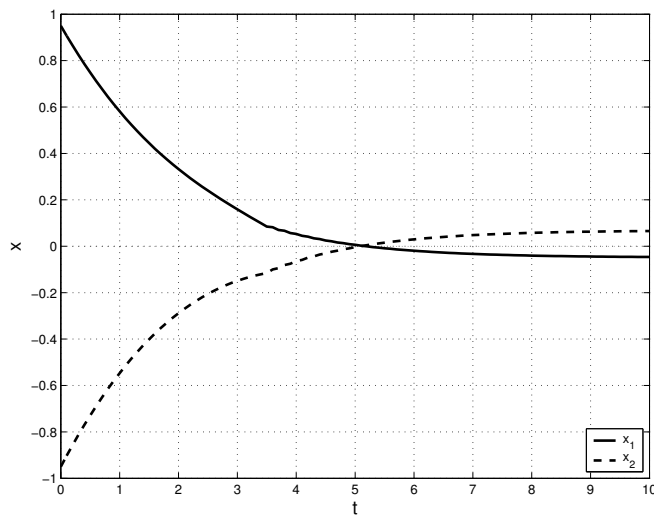


Fig. 1. State trajectories of the example system starting from the initial condition $x_0 = [0.95 - 0.95]$.

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